Euler Equation Examples

1. Solve $2t^2y'' + 3ty' - 15y = 0$

Soln:

$$t^2y'' + \frac{3ty'}{2} - \frac{15}{2}y = 0$$

$$r^2 + (\alpha - 1)r - \beta = 0$$

In this case $\alpha = \frac{3}{2}$ and $\beta = \frac{-15}{2}$

$$r^2 + \frac{r}{2} - \frac{15}{2} = 0$$

$$2r^2 + r - 15 = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1 - 4(2)(-15)}}{4}$$

$$= \frac{-1 \pm \sqrt{121}}{4}$$

$$= \frac{-1 \pm 11}{4}$$

$$= \frac{10}{4} \text{ or } \frac{-12}{4}$$

$r_1 = \frac{5}{2}$ $\quad$ $r_2 = -3$

$y_1 = t^{r_1} = t^{5/2}$ $\quad$ $y_2 = t^{r_2} = t^{-3}$ $\quad$ $y = c_1y_1 + c_2y_2 = c_1t^{5/2} + c_2t^{-3}$

2. Solve $t^2y'' - 7ty' + 16y = 0$

Soln:
$\alpha = -7, \ \beta = 16$

$r^2 + (\alpha - 1)r + \beta = 0$
$r^2 - 8r + 16 = 0$
$(r-4)^2 = 0$
$r = 4$

$y_1 = t^r = t^4$ $\qquad$ $y_2 = (\ln t)t^r = (\ln t)(t^4)$

$y = c_1 y_1 + c_2 y_2$
$= c_1 t^4 + c_2 (\ln t)(t^4)$

3. Solve $t^2y'' + 3ty' + 4y = 0$

Soln:
$\alpha = 3, \ \beta = 4$

$r^2 + (\alpha - 1)r + \beta = 0$
$r^2 + 2r + 4 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 16}}{2} = \frac{-2 \pm \sqrt{-12}}{2} = \frac{-2 \pm 2\sqrt{-3}}{2} = -1 \pm \sqrt{3}i$$

$\lambda = -1, \ u = \sqrt{3}$

$r = -1 \pm \sqrt{3}i$
We will only use $r = -1 + \sqrt{3}i$.

$$y = t^r = t^{-1+\sqrt{3}i} = \frac{t^{\sqrt{3}i}}{t}$$

$t = e^{\ln t}$
$t^{\sqrt{3}i} = e^{\sqrt{3}i \ln t}$

$$\frac{t^{\sqrt{3}i}}{t} = \frac{e^{\sqrt{3}i \ln t}}{t}$$

$$e^{\sqrt{3}i\ln t} = \cos(\sqrt{3}\ln t) + i\sin(\sqrt{3}\ln t)$$

$$\frac{\cos(\sqrt{3}\ln t) + i\sin(\sqrt{3}\ln t)}{t}$$

$$y_1 = \frac{\cos(\sqrt{3}\ln t)}{t}, \quad y_2 = \frac{\sin(\sqrt{3}\ln t)}{t}$$

$$y = \frac{c_1\cos(\sqrt{3}\ln t)}{t} + \frac{c_2\sin(\sqrt{3}\ln t)}{t}$$

4. Solve $t^2y'' - 3ty' + 4y = 0$

Soln:

$\alpha = -3, \ \beta = 4$

$r^2 + (\alpha-1)r + \beta = 0$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2$

$y_1 = t^r \qquad y_2 = \ln(t)t^r$

$\quad = t^2 \qquad\quad = \ln(t)t^2$

$y = c_1t^2 + c_2\ln(t)t^2$

5. Solve $t^2y'' + 2ty' + 0.25y = 0$

Soln:

$\alpha = 2, \ \beta = 0.25$

$r^2 + (\alpha-1)r + \beta = 0$

$r^2 + r + 0.25 = 0$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1-1}}{2}$$

$$= \frac{-1}{2}$$

$y_1 = t^r \qquad y_2 = \ln(t)t^r \qquad y = c_1t^{-1/2} + c_2\ln(t)t^{-1/2}$

$\quad = t^{-1/2} \qquad\quad = \ln(t)t^{-1/2}$